RATIONALIZATION

Rationalization is a process where an irrational number is eliminated in the denominator. Sometimes when one deals with finding limits in Calculus, we rationalize the numerator. We will also deal with questions where we need to rationalize both numerator and denominator.

One concept we discuss first is the finding the Conjugate of an expression. Here are some examples. Two binomials are Conjugates when they have the same term but opposite Signs in the middle.

The conjugate of:

a) $7-\sqrt{2}$ is $7+\sqrt{2}$
b) $8+\sqrt{3}$ is $8-\sqrt{3}$
c) $-8+\sqrt{5}$ is $-8-\sqrt{5}$
d) $3-2\sqrt{3}$ is $3+2\sqrt{3}$
e) $2-\sqrt{x}$ is $2+\sqrt{x}$
f) $\sqrt{7-x}-2$ is $\sqrt{7-x}+2$
g) $1-\sqrt{4-x}$ is $1+\sqrt{4-x}$

when we multiply a binomial with its Conjugate, we get a simplified expression.

Let us find the Product of the above binominals with the Conjugates. Before that please remember

$$(a+b)(a-b) = a^2 - b^2$$

$$(7-\sqrt{2})(7+\sqrt{2})=(7)^2-(\sqrt{2})^2=49-2=47$$

$$(8+\sqrt{3})(8-\sqrt{3})=(8)^2-(\sqrt{3})^2=64-3=61$$

$$(-8+\sqrt{5})(-8-\sqrt{5})=(-8)^2-(\sqrt{5})^2=64-5=59$$

$$(3-2\sqrt{3})(3+2\sqrt{3})=(31^2-(2\sqrt{3})^2=9-12=-3$$

$$(2-\sqrt{5}x)(2+\sqrt{5}x)=(2)^2-(\sqrt{5}x)^2=4-5x^2$$

$(\sqrt{7-x}-2)(\sqrt{7-x}+2) = (\sqrt{7-x})^2 - 2^2 = 7-x-4$
$= 3 - x$

$(1-\sqrt{4-x})(1+\sqrt{4-x}) = (1)^2 - (\sqrt{4-x})^2$
$= 1-(4-x) = 1-4+x = -3+x$

Here, we are only discussing Rationalization of expression with square roots. Also, to Rationalize the radical expression √2, we multiply √2 by √2

⇨(√2) (√2) = 2.

We give some examples below.

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Example: Rationalize $\frac{1}{\sqrt{5}}$

The denominator is √5. so, we multiply the numerator and denominator by √5.

$$\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}}$$

Example: Rationalize $\frac{\sqrt{8}}{\sqrt{24}}$

Here, we do not rationalize the denominator by multiplying the numerator and denominator by $\sqrt{24}$ because we can break down the radical $\sqrt{24}$ unlike $\sqrt{5}$ above.

So, $\sqrt{24} = \sqrt{4\times 6} = 2\sqrt{6}$

$$\Rightarrow \frac{\sqrt{8}}{\sqrt{24}} = \frac{8}{2\sqrt{6}}$$

To rationalise the denominator, we multiply the numerator and Denominator by √6.

$$\frac{8}{2\sqrt{6}}\frac{(\sqrt{6})}{(\sqrt{6})} = \frac{8\sqrt{6}}{12} = \frac{2\sqrt{6}}{3}$$

Example: Rationalize $\frac{1}{3-\sqrt{5}}$

We multiply the numerator and Denominator by $3+\sqrt{5}$

$$\frac{1}{3-\sqrt{5}}\;\frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{3+\sqrt{5}}{(3)^2-(\sqrt{5})^2} = \frac{3+\sqrt{5}}{9-5} = \frac{3+\sqrt{5}}{4} = \frac{3}{4} + \frac{1}{4}\sqrt{5}$$

4) $\frac{6}{\sqrt{8}-3}$

We multiply the Numerator and Denominator by $\sqrt{8} + 3$.

$$\frac{6}{\sqrt{8}-3}\;\frac{\sqrt{8}+3}{\sqrt{8}+3} = \frac{6(\sqrt{8}+3)}{(\sqrt{8})^2-(3)^2} = \frac{6\sqrt{8}+18}{8-9} = \frac{6\sqrt{8}+18}{8-9} = \frac{6\sqrt{8}+18}{-1}$$

$$= -6\sqrt{8} - 18$$

Example: $\frac{3\sqrt{6}-7\sqrt{2}}{5\sqrt{6}-2\sqrt{2}}$

Solution: We multiply the numerator and Denominator by

$5\sqrt{6} + 2\sqrt{2}$

$$\frac{(3\sqrt{6}-7\sqrt{2})}{(5\sqrt{6}-2\sqrt{2})}\;\frac{(5\sqrt{6}+2\sqrt{2})}{(5\sqrt{6}+2\sqrt{2})} = \frac{(3\sqrt{6}-7\sqrt{2})(5\sqrt{6}+2\sqrt{2})}{(5\sqrt{6})^2-(2\sqrt{2})^2}$$

$$= \frac{(3\sqrt{6})(5\sqrt{6})+(3\sqrt{6})(2\sqrt{2})-(7\sqrt{2})(5\sqrt{6})-7\sqrt{2}(2\sqrt{2})}{150-8}$$

$$= \frac{90-6\sqrt{12}-35\sqrt{12}+28}{142} = \frac{90-41\sqrt{12}+28}{142} = \frac{90-41\sqrt{3\times4}+28}{142}$$

$$= \frac{90-41(2\sqrt{3})+28}{142} = \frac{118-82\sqrt{3}}{142} = \frac{2(59-41\sqrt{3})}{142} = \frac{59-41\sqrt{3}}{142}$$

$$= \frac{59-4\sqrt{3}}{71} = \frac{59}{71} - \frac{4\sqrt{3}}{71}$$

Example: $x = 3 + \sqrt{8}$, find the value of $x^2 + \frac{1}{x^2}$

Solution: $x = 3 + \sqrt{8}$

So, $\frac{1}{x} = \frac{1}{3+\sqrt{8}} = \frac{1}{3+\sqrt{8}}\;\frac{3-\sqrt{8}}{3-\sqrt{8}} = \frac{3-\sqrt{8}}{(3)^2-(\sqrt{8})^2} = \frac{3-\sqrt{8}}{9-8} = 3-\sqrt{8}$

So, $x^2 + \frac{1}{x^2} = (3+\sqrt{8})^2 + (3-\sqrt{8})^2$

We use the formula, $(a+b)^2 - (a-b)^2 = 2(a^2 + b^2)$

Here, a=3 and b=$\sqrt{8}$

So, $(3+\sqrt{8})^2 + (3-\sqrt{8})^2 = 2(3^2 + (\sqrt{8})^2) = 2(9+8) = 34$

Example: If $\frac{3+\sqrt{5}}{3-\sqrt{5}} = a + b\sqrt{5}$ and a and b are rational numbers, find the value of a and b.

Solution: The Conjugate of $3-\sqrt{5}$ is $3+\sqrt{5}$

We multiply the numerator and Denominator by $3+\sqrt{5}$

So, $\frac{3+\sqrt{5}}{3-\sqrt{5}} \frac{(3+\sqrt{5})}{(3+\sqrt{5})} = \frac{(3+\sqrt{5})^2}{(3)^2-(\sqrt{5})^2} = \frac{9+(\sqrt{5})^2+2(3)(5)}{9-5} = \frac{9+5+6\sqrt{5}}{4}$

$\frac{14+6\sqrt{5}}{4} = \frac{2(7+3\sqrt{5})}{4} = \frac{7+3\sqrt{5}}{2} = \frac{7}{2} + \frac{3}{2}\sqrt{5}$

So, a $= \frac{7}{2}$ and b $= \frac{3}{2}$

Example: Rationalize $\frac{\sqrt{2+h}-2}{h}$ by rationalizing the numerator.

We Rationalise the numerator $\sqrt{2+h} - 2$

The Conjugate of $\sqrt{2+h} - 2$ is $\sqrt{2+h} + 2$

We multiply the numerator and denominator by $\sqrt{2+h} + 2$

$\frac{\sqrt{2+h}-2}{h} \frac{(\sqrt{2+h}+2)}{(\sqrt{2+h}+2)} = \frac{(\sqrt{2+h})^2-(2)^2}{h(\sqrt{2+h}+2)} = \frac{2+h-4}{h[\sqrt{2+h}+2]} = \frac{h-2}{h[\sqrt{2+h}+2]}$

Example: Rationalize $\frac{x-4}{\sqrt{x}-\sqrt{8-x}}$ by rationalizing the denominator.

Conjugate of denominator $\sqrt{x} - \sqrt{8-x}$ is $\sqrt{x} + \sqrt{8-x}$

We multiply both numerator and denominator by $\sqrt{x} + \sqrt{8-x}$

$$\frac{x-4}{\sqrt{x}-\sqrt{8-x}}\,\frac{(\sqrt{x}+\sqrt{8-x})}{(\sqrt{x}+\sqrt{8-x})}=\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{(\sqrt{x})^2-(\sqrt{8-x})^2}$$

$$=\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{x-(8-x)}=\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{x-8+x}=\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2x-8}$$

$$=\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2(x-4)}=\frac{\sqrt{x}+\sqrt{8-x}}{2}$$

<u>Example:</u> Rationalize $\frac{\sqrt{7-x}-2}{1-\sqrt{4-x}}$ by rationalizing the numerator and denominator.

<u>Solution:</u> Here we rationalise both numerator and Denominator.

Conjugate of $\sqrt{7-x}$ -2 is $\sqrt{7-x}$ +2

Conjugate of 1-$\sqrt{4-x}$ is 1+$\sqrt{4-x}$

$$=\frac{\sqrt{7-x}-2}{1-\sqrt{4-x}}\cdot\frac{\sqrt{7-x}+2}{\sqrt{7-x}+2}\cdot\frac{1+\sqrt{4-x}}{1+\sqrt{4-x}}$$

$$=\frac{\sqrt{7-x}-2}{1-\sqrt{4-x}}\cdot\frac{\sqrt{7-x}+2}{1+\sqrt{4-x}}\cdot\frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}$$

$$=\frac{(\sqrt{7-x})^2-(2)^2}{(1)^2-(\sqrt{4-x})^2}\cdot\frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}=\frac{7-x-4}{1-(4-x)}\cdot\frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}$$

$$=\frac{3-x}{(-3+x)}\,\frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}=\frac{3-x}{-(3-x)}\cdot\frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}$$

$$=\frac{-(1+\sqrt{4-x})}{\sqrt{7-x}+2}$$

<u>Example:</u> Let $f(x)=\frac{9x^2-90x+81}{9-3\sqrt{x}}$. $f(9)$ is not defined because when you substitute $x=9$ we get $\frac{0}{0}$ which is undefined. Rationalize $f(x)$ by rationalizing the denominator and get value of $f(9)$.

Solution: $f(x) = \frac{9x^2 - 90x + 81}{9 - 3\sqrt{x}} = \frac{9(x-1)(x-9)}{3(3-\sqrt{x})} = \frac{3(x-1)(x-9)}{3-\sqrt{x}}$

We rationalize f(x) by multiplying numerator and denominator $3 + \sqrt{x}$.

$$\frac{3(x-1)(x-9)}{3-\sqrt{x}} . \frac{3+\sqrt{x}}{3+\sqrt{x}} = \frac{3(x-1)(x-9)(3+\sqrt{x})}{(3)^2-(\sqrt{x})^2} = \frac{3(x-1)(x-9)(3+\sqrt{x})}{9-x}$$

$$= \frac{3(x-1)(x-9)(3+\sqrt{x})}{-(x-9)} = -3(x-1)(3+\sqrt{x})$$

So, $f(x) = -3(x-1)(3+\sqrt{x})$

So, $f(x) = -3(9-1)(3+\sqrt{9}) = = -3(8)(6) = -144$