RATIONAL EQUATIONS

A rational equation is an expression of the form

$$\frac{f(x)}{g(x)} = 0,\ g(x) \neq 0$$

If f(x) and g(x) are polynomial functions with no common factors, then the zeroes of $f(x)$ are the solution of the equation. To solve a rational equation, you do not have a put the rational function in the form, $\frac{f(x)}{g(x)}$ we get rid of the fractions by multiplying each side by the LCD (Least Common Denominator). Please go through the topic "Least Common Multiple (LCD) OR Last Common Multiple of rational expressions" before you go further.

After this process, we may get solutions that are not solutions of the original equation. There are extraneous solutions. For this reason, we must check each solution of the resulting equation by substituting it in the original solution.

Example: Solve $x+\frac{5}{x}= 6$

Solution

$x+\frac{5}{x} = 6$ can be written as $\frac{x}{1} + \frac{5}{x} =6$

The LCD $(1, x)=x$

we multiply each side by x

$$\left(\frac{x}{1} + \frac{5}{x} = 6\right)x \Rightarrow x^2 + 5 = 6\ x$$

$x^2 - 6x + 5 = 0 \Rightarrow\ (x-1)(x-5) = 0 \Rightarrow x = 1\ or\ 5$

Now, we check these two solutions by substituting

x = 1 and x = 5 in $x + \frac{5}{x} = 6$

For x = 1 $1+\frac{5}{1}=6$ $1+5=6$ ⇨ $6=6$ ⇨ which is true

For x = 5

$5 + \frac{5}{5} = 6 \Rightarrow 5 + 1 = 6 \Rightarrow 6 = 6$ which is true.

So, both $x = 1$ *and* $x = 5$ are not extraneous solutions

$x = 1$ *and* $x = 5$ are the real solutions of the Rational equation $x + \frac{5}{x} = 6$

Example:

Solve $\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{x^2-4x+3}$

Solution After factoring $x^2 - 4x + 3$ to $(x-1)(x-3)$

we rewrite above equation as

$$\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{(x-1)(x-3)} \qquad \textbf{(1)}$$

Now, we find LCD $[(x-1), (x-3), (x-1)(x-3)]$

which is $(x-1)(x-3)$

we multiply education (1)by $(x-1)(x-3)$

$$\left(\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{(x-1)(x-3)}\right)(x-1)(x-3)$$

$$2x(x-3) + (x-1) = 2 \qquad \textbf{(2)}$$

We assume you understand how you get the above results. The Multiplications carried out are as follows.

$$\boxed{\begin{array}{l} \frac{2x}{x-1}(x-1)(x-3) = 2x(x-3) \\ \frac{1}{x-3}(x-1)(x-3) = x-1 \\ \frac{2}{(x-1)(x-3)}(x-1)(x-3) = 2 \end{array}}$$

So, equation (2) becomes $2x^2 - 6x + x - 1 = 2$

⇨ $2x^2 - 5x - 3 = 0$

$(2x+1)(x-3) = 0$ ⇨ $x = \frac{-1}{2}$ or $x = 3$

Now, let us check if these two solutions are extraneous by substituting in the original equation.

$\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{x^2-4x+3}$

For x $= \frac{-1}{2}$

$\frac{2(\frac{-1}{2})}{\frac{-1}{2}-1} + \frac{1}{\frac{-1}{2}-3} = \frac{2}{\left(\frac{-1}{2}\right)^2 - 4\left(\frac{-1}{2}\right)+3}$

$\frac{2}{3}$ - $\frac{2}{7} = \frac{8}{21}$ ⇨$\frac{8}{21} = \frac{8}{21}$ which is true

So, x $= \frac{-1}{2}$ is not an extraneous Solution.

<u>**Now, let us check $x = 3$**</u>

We note that when we substitute $x = 3$ in

$\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{x^2-4x+3}$

we get a zero denominator because of the presence of $\frac{1}{x-3}$ and $\frac{2}{x^2-4x+3}$

$\frac{2(3)}{3-1}+\frac{1}{3-3}=\frac{2}{3^2-4(3)+3}$ ⇨ $\frac{6}{2}+\frac{1}{0}=\frac{2}{0}$

Both $\frac{1}{0}$ and $\frac{2}{0}$ are undefined.

So, we conclude that $x = 3$ is an extraneous Solution

Example $\frac{x-3}{x}+\frac{3}{x+2}+\frac{6}{x^2+2x}=0$

Solution $\frac{x-3}{x}+\frac{3}{x+2}+\frac{6}{x(x+2)}=0$ **(1)**

$LCD[x,(x+2),x(x+2)] = x(x+2)$

We multiply equation (1) by $x(x+2)$

$$\left[\frac{x-3}{x}+\frac{3}{x+2}+\frac{6}{x(x+2)}=0\right]x(x+2)$$

$(x-3)(x+2)+3x+6=0$

$x^2-x-6+3x+6=0$

$x^2+2x=0$

$x(x+2)=0$

$x = 0$ or $x = -2$

Now, we substitute $x = 0$ and $x = -2$ in

$\frac{x-3}{x}+\frac{3}{x+2}+\frac{6}{x^2+2x}$ to check our solutions

$x = 0$ gives a zero denominatorbecause of the presence of $\frac{x-3}{x}$

$x = 2$ gives a zero denominator because of the presence of $\frac{3}{x+2}$

So, both $x = 0$ and $x = -2$ are extraneous Solutions.

Example $\frac{1}{6a^2}+\frac{1}{6a}=\frac{1}{a^2}$

Solution LCD $(6a^2, 6a, a^2)= 6a^2$

We multiply the equation by $6a^2$

$\left(\frac{1}{6a^2}+\frac{1}{6a}=\frac{1}{a^2}\right)6a^2$

$1+\mathrm{a}=6 \Rightarrow \mathrm{a}=5$

Check for a =5

$$\frac{1}{6(5)^2}+\frac{1}{6(5)}=\frac{1}{5^2}$$

$$\frac{1}{150}+\frac{1}{30}=\frac{1}{25}$$

$$\frac{6}{150}=\frac{1}{25} \Rightarrow \frac{1}{25}=\frac{1}{25}$$

So, $\mathrm{a}=5$ is a Solution

Example $\frac{1}{\mathrm{n}-8}-1=\frac{7}{\mathrm{n}-8}$

$$\Rightarrow \frac{1}{\mathrm{n}-8}-\frac{1}{1}=\frac{7}{\mathrm{n}-8} \qquad \textbf{(1)}$$

LCD $(\mathrm{n}-8, 1, \mathrm{n}-8)=\mathrm{n}-8$

Multiply (1) by $\mathrm{n}-8$

$\left(\frac{1}{\mathrm{n}-8}-\frac{1}{1}=\frac{7}{\mathrm{n}-8}\right)(\mathrm{n}-8)$

$1-(\mathrm{n}-8)=7 \quad \Rightarrow 1-\mathrm{n}+8=7 \Rightarrow 9-\mathrm{n}=7 \quad \Rightarrow \mathrm{n}=2$

Check for n=2

Substitute n=2 in $\frac{1}{\mathrm{n}-8}-1=\frac{7}{\mathrm{n}-8}$

$\frac{1}{2-8}-1=\frac{7}{2-8} \quad \Rightarrow \frac{1}{-6}-1=\frac{7}{-6} \Rightarrow \frac{7}{-6}=\frac{7}{-6}$

So, $\mathrm{x}=2$ is a Solution.

Example: $\frac{x+5}{x^2-2x} - 1 = \frac{1}{x^2-2x}$

Solution $\frac{x+5}{x(x-2)} - \frac{1}{1} = \frac{1}{x(x-2)}$ (1)

$\text{LCD}\,[\,x(x-2), 1, x(x-2)] = x(x-2)$

Multiply (1) by $x(x-2)$

$$\left[\frac{x+5}{x(x-2)} - \frac{1}{1} = \frac{1}{x(x-2)}\right] x(x-2)$$

$x + 5 - x(x-2) = 1 \Rightarrow x + 5 - x^2 + 2x = 1$

$-x^2 + 3x + 4 = 0 \Rightarrow x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0 \Rightarrow x = 4 \text{ or } x = -1$

Check for $x = 4$

Substitute $x = 4$ in the equation

$\frac{x+5}{x^2-2x} - 1 = \frac{1}{x^2-2x}$

$\frac{4+5}{4^2-2(4)} - 1 = \frac{1}{4^2-2(4)} \Rightarrow \frac{9}{8} - 1 = \frac{1}{8} \Rightarrow \frac{1}{8} = \frac{1}{8}$

So, $x = 4$ is a Solution

Check for $x = -1$

Substitute $x = -1$ in the equation

$\frac{x+5}{x^2-2x} - 1 = \frac{1}{x^2-2x}$

$\frac{-1+5}{(-1)^2-2(-1)} - 1 = \frac{1}{(-1)^2-2(-1)} \Rightarrow \frac{4}{3} - 1 = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{1}{3}$

So, $x = -1$ is a Solution

Example: Solve $\frac{x-3}{2x+10} + 2x - 12 = \frac{x^2+3x-18}{2x+10}$

Solution First we factor $\frac{x-3}{2(x+5)} + \frac{2(x-6)}{1} = \frac{(x+6)(x-3)}{2(x+5)}$ **(1)**

$LCD[2(x+5), 1, 2(x+5)] = 2(x+5)$

*You could do this equation without factoring too

Multiply equation (1) by $2(x+5)$

$$\left(\frac{x-3}{2(x+5)} + \frac{2(x-6)}{1} = \frac{(x+6)(x-3)}{2(x+5)}\right)2(x+5)$$

$(x-3) + 4(x-6)(x+5) = (x+6)(x-3)$

$x - 3 + 4(x^2 - x - 30) = x^2 + 3x - 18$

$x - 3 + 4x^2 - 4x - 120 = x^2 + 3x - 18$

$4x^2 - 3x - 123 = x^2 + 3x - 18$

$3x^2 - 6x - 105 = 0$

$3(x^2 - 2x - 35) = 0 \Rightarrow x^2 - 2x - 35 = 0$

$(x-7)(x+5) = 0 \Rightarrow x = 7$ or $x = -5$

Check for $x = 7$

Substitute $x = 7$ in $\frac{x-3}{2x+10} + 2x - 12 = \frac{x^2+3x-18}{2x+10}$

$$\frac{7-3}{2(7)+10} + 2(7) - 12 = \frac{(7)^2 + 3(7) - 18}{2(7)+10}$$

$\frac{4}{24} + 2 = \frac{52}{24} \Rightarrow \frac{52}{24} = \frac{52}{24}$

So, $x = 7$ is a solution .

Check for $x = -2$

Substitute $x = -2$ *in* $\frac{x-3}{2x+10} + 2x - 12 = \frac{x^2+3x-18}{2x+10}$

we will get a term with 0 denominator because of the presence of

$\frac{x-3}{2x+10}$ and $\frac{x^2+3x-18}{2x+10}$ as $2(-5)+10=-10+10=0$

So, $x=-2$ is an extraneous solution .